



IE463 – Chapter 2

Time Value of Money (Money-Time Relationships)



Objective

Given a cash flow (or series of cash flows) occurring at some point in time, the objective is to find its equivalent value at another point in time considering the time value of money.

Time Value of Money

- \$100 today or \$120 one year from now?
- Why does money have a time value?
- What affects money's time value (i.e. interest rate)?

Two Ways for Calculating the Time-Value of Money:

1. Simple Interest
2. Compound Interest

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Simple Interest

The amount of interest earned (or paid) is directly proportional to the principal of the loan, the number of interest periods for which the principal is committed, and the interest rate per period.

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Simple Interest Example

Borrow \$1000 for 5 yrs. at 12% interest/yr.

	(1)	(2) = $P \times 12\%$	(3)=(1)+(2)
Period (year)	Amount Owed at Beginning of Period	Interest Amount for Period	Amount Owed at End of Period
1	\$ 1,000	\$ 120	\$ 1,120
2	\$ 1,120	\$ 120	\$ 1,240
3	\$ 1,240	\$ 120	\$ 1,360
4	\$ 1,360	\$ 120	\$ 1,480
5	\$ 1,480	<u>\$ 120</u>	\$ 1,600
		\$ 600	

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Compound Interest

The amount of interest earned (or paid) per interest period depends on the remaining principal of the loan plus any unpaid interest charges.

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Compound Interest Example

borrow \$1000 for 5 yrs. at 12% interest/yr.

	(1)	(2) = (1) x 12%	(3)=(1)+(2)
Period (year)	Amount Owed at Beginning of Period	Interest Amount for Period	Amount Owed at End of Period
1	\$ 1,000.00	\$ 120.00	\$ 1,120.00
2	\$ 1,120.00	\$ 134.40	\$ 1,254.40
3	\$ 1,254.40	\$ 150.53	\$ 1,404.93
4	\$ 1,404.93	\$ 168.59	\$ 1,573.52
5	\$ 1,573.52	<u>\$ 188.82</u>	\$ 1,762.34
		\$ 762.34	

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Concept of Equivalence

- To compare alternatives that provide the same service over extended periods of time when interest is involved, we must reduce them to an equivalent basis that is dependent on:
 - the interest rate,
 - the amounts of money involved,
 - the timing of the monetary receipts or expenses.
- If two alternatives are economically equivalent, then they are equally desirable.

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Equivalence Example

- Amount of Principal = \$ 8000
- Interest rate = 10% /yr.
- Duration of the loan = 4 years

Consider two different payment plans to see the meaning of equivalence.

Plan1: At the end of each year pay \$2000 principal plus interest due.

Plan4: Pay principal and interest in one payment at the end of four years.

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Plan1: At the end of each year pay \$2000 principal plus interest due

(1) (2) (3)=0.1x(2) (4)=(2)+(3) (5) (6)=(3)+(5)

Year	Amount owed at beginning of year	Interest Accrued for year	Total Money Owed at end of year	Principal payment	Total End-of-Year payment (Cash Flow)
1	\$ 8000	\$ 800	\$ 8800	\$ 2000	\$ 2800
2	6000	600	6600	2000	2600
3	4000	400	4400	2000	2400
4	2000	200	2200	2000	2200
	\$ 20,000 total dollar-years	\$ 2000 total interest		\$ 8000	\$ 10,000 total amount repaid

Plan4: Pay principal and interest in one payment at the end of four years

(1) (2) (3)=0.1x(2) (4)=(2)+(3) (5) (6)=(3)+(5)

Year	Amount owed at beginning of year	Interest Accrued for year	Total Money Owed at end of year	Principal payment	Total End-of-Year payment (Cash Flow)
1	\$ 8000	\$ 800	\$ 8800	\$0	\$0
2	8800	880	9680	0	0
3	9680	968	10,648	0	0
4	10,648	1065	11,713	8000	11,713
	\$ 37,130 total dollar-years	\$ 3713 total interest		\$ 8000	\$ 11,713 total amount repaid

Comparison of Plans 1 & 4

Summing the amount owed at the beginning of each year yields the amount of money "rented" in dollar-years.

Plan	Dollar-years	Total Interest Paid	Ratio of Total Interest to Dollar-Years
1	\$ 20,000	\$2,000	0.10
4	\$ 37,130	\$3,713	0.10

Equivalence is established when total interest paid, divided by dollar-years of borrowing, is a constant ratio among financing plans.

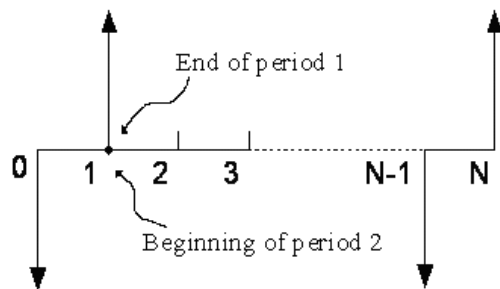
Principles of Equivalence

- Equivalent cash flows have the same economic value at the same point in time.
- Conversion of a cash flow to its equivalent, at another point in time, must reflect the interest rate(s) in effect for each period between the cash flows.
- Equivalence between receipts and disbursements: the interest rate that sets the receipts equivalent to the disbursements is the actual interest rate.
- Economic equivalence is established, in general, when we are indifferent between a future payment, or series of payments, and a present sum of money.

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Cash Flow Diagrams (CFDs)

+ Receipts (Revenues, Benefits, Savings, Cash inflows)



– Disbursements (Expenses, Cash outflows, Expenditures)

Use a consistent viewpoint

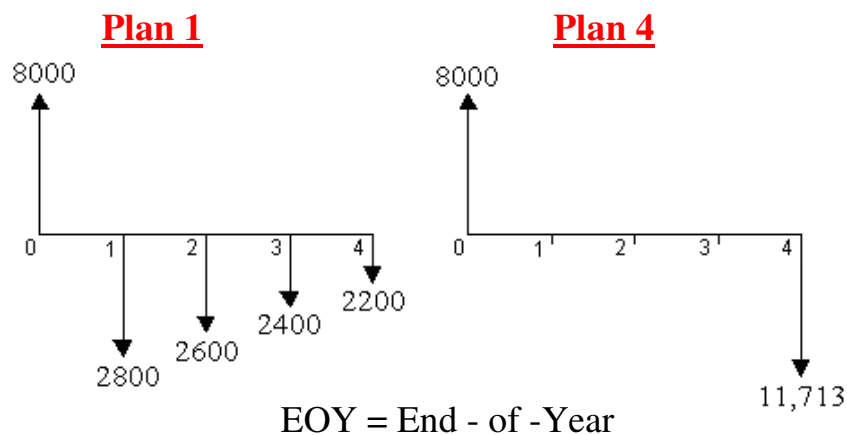
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Cash Flows of Plans 1 & 4 from Borrower's viewpoint

	Plan 1	Plan 4
EOY	Cash Flows	Cash Flows
0	\$ 8000	\$8000
1	– 2800	0
2	– 2600	0
3	– 2400	0
4	– 2200	– 11,713

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Cash Flow Diagrams of Plans 1 & 4 from Borrower's viewpoint



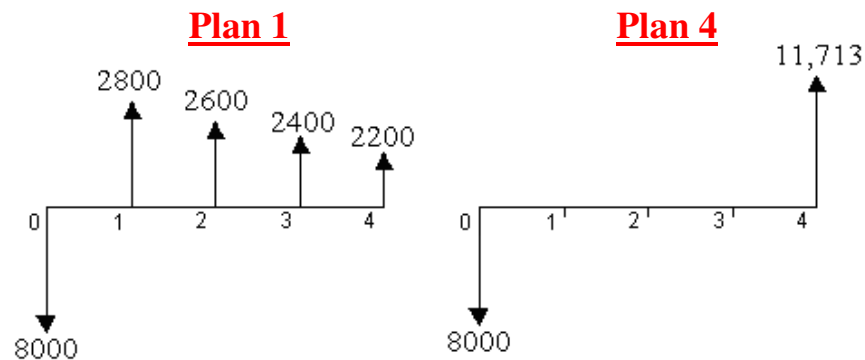
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Cash Flows of Plans 1 & 4 from Lender's viewpoint

	Plan 1	Plan 4
EOY	Cash Flows	Cash Flows
0	– \$ 8000	– \$8000
1	2800	0
2	2600	0
3	2400	0
4	2200	11,713

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Cash Flow Diagrams of Plans 1 & 4 from Lender's viewpoint



EOY = End - of -Year

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Interest Formulas Relating to Single Cash Flows

1. How to find the Future Equivalent value of a Present Sum of Money
2. How to find the Present Equivalent value of a Future Sum of Money

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How to find the Future Equivalent value of a Present Sum of Money

Period	Amount at Beginning of Period	Interest Earned During the Period	Amount at end of Period
1	P	P(i)	P(1+i)
2	P(1+i)	P(1+i)(i)	P(1+i)²
3	P(1+i) ²	P(1+i) ² (i)	P(1+i)³
⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮
N – 1	P(1+i) ^{N-2}	P(1+i) ^{N-2} (i)	P(1+i)^{N-1}
N	P(1+i) ^{N-1}	P(1+i) ^{N-1} (i)	P(1+i)^N

$F_N = P(1+i)^N$ where,

P = a present sum of money
i = effective interest /period
N = number of periods
F = a future sum of money

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Single Payment Compound Amount Factor

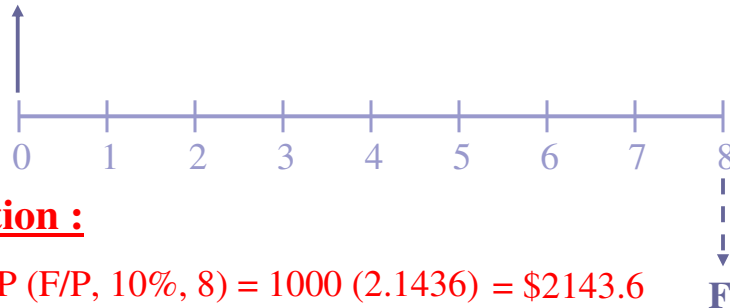
- The quantity $(1+i)^N$ in the equation is commonly called the *single payment compound amount factor*.
- Numerical values for this factor have been calculated for a wide range of values of i and N , and they are available in the tables of Appendix C
 - $(1+i)^N = (F/P, i\%, N)$
 - $F = P (F/P, i\%, N)$

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Example for finding F given P

A firm borrows \$1000 for eight years at $i = 10\%$. How much must it repay in a lump sum at the end of eighth year?

$P = 1000$



Solution :

$$F_8 = P (F/P, 10\%, 8) = 1000 (2.1436) = \$2143.6$$

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How to find the Present Equivalent value of a Future Sum of Money

$$F = P(1+i)^N$$

$$\rightarrow P = F [1 / (1+i)^N] = F(1+i)^{-N}$$

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Single Payment Present Worth Factor

- The quantity $(1+i)^{-N}$ in the equation is commonly called the *single payment present worth factor*.
- Numerical values for this factor have been calculated for a wide range of values of i and N , and they are available in the tables of Appendix C.
 - $(1+i)^{-N} = (P / F, i\%, N)$
 - $P = F (P / F, i\%, N)$

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Example for finding P given F

How much would you have to deposit now into an account paying 10% interest per year in order to have \$1,000,000 in 40 years?

Assumptions:

Constant interest rate; no additional deposits or withdrawals

Solution:

$$P = \$1000,000 (P/F, 10\%, 40) = \$22,100$$

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Interest Formulas Relating to Annuities

Annuities: Uniform end-of-period amounts

1. Given a uniform end-of-period series, and find its equivalent present value
2. Given a present sum of money find its equivalent uniform end-of-period series
3. Given a uniform end-of-period series find its equivalent future value
4. Given a future sum of money find its equivalent uniform end-of-period series

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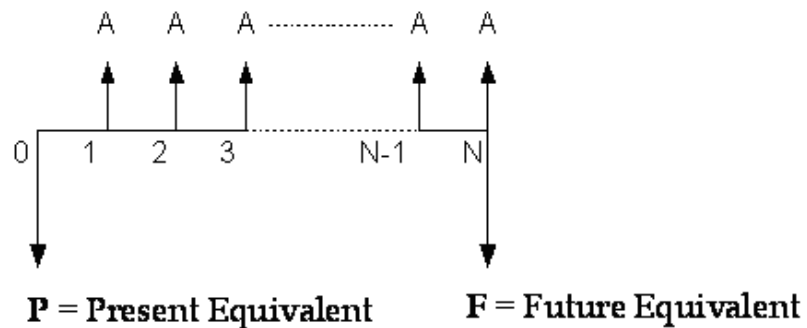
Timing relationships for P, A and F

- P (present equivalent value) occurs one interest period before the first A (uniform amount).
- F (future equivalent value) occurs at the same time as the last A, and N periods after P.
- A (annual equivalent value) occurs at the end of periods 1 through N, inclusive.

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Timing relationships for P, A and F

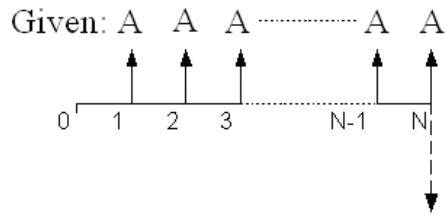
A = Uniform Amounts



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Finding F When Given A

Given a uniform series of payments, A, for N periods, how might we determine the equivalent future sum, F, at $t = N$?



We could treat each A as a separate present sum:

$$F = A(1+i)^{N-1} + A(1+i)^{N-2} + A(1+i)^{N-3} + \dots + A(1+i)^1 + A(1+i)^0$$

This equation can be reduced to: $F = A \left[\frac{(1+i)^N - 1}{i} \right]$

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Uniform Series Compound Amount Factor

■ $\left[\frac{(1+i)^N - 1}{i} \right]$ is the *uniform series compound amount factor*

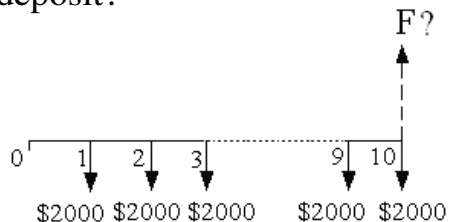
■ $\left[\frac{(1+i)^N - 1}{i} \right] = (F/A, i\%, N)$

■ $F = A (F/A, i\%, N)$

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Example for finding F given A

Assume you make 10 equal annual deposits of \$2,000 into an account paying 5% per year. How much is in the account just after the 10th deposit?



Solution

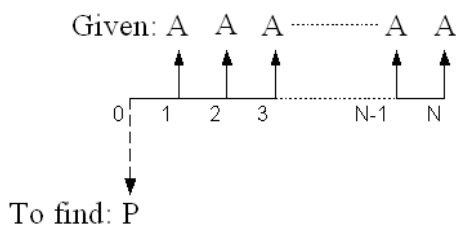
12.5779

$$F = \$2,000 (F/A, 5\%, 10) = \$25,156$$

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Finding P When Given A

If we wish to withdraw a uniform sum (A) for N years from an account paying $i\%$ interest, how much must we deposit now (P)?



We could treat each A as a separate future sum:

$$P = A(1+i)^{-1} + A(1+i)^{-2} + A(1+i)^{-3} + \dots + A(1+i)^{-(N-1)} + A(1+i)^{-N}$$

$$\text{or } P(1+i)^N = A \left[\frac{(1+i)^N - 1}{i} \right] \Rightarrow P = A \left[\frac{(1+i)^N - 1}{i(1+i)^N} \right]$$

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Uniform Series Present Worth Factor

■ $\left[\frac{(1+i)^N - 1}{i(1+i)^N} \right]$ is the *uniform series present worth factor*

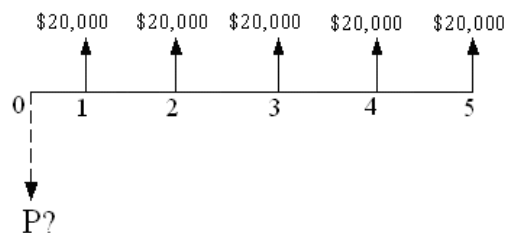
■ $\left[\frac{(1+i)^N - 1}{i(1+i)^N} \right] = (P/A, i\%, N)$

■ $P = A (P/A, i\%, N)$

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Example for finding P given A

If a certain machine undergoes a major overhaul now, its output can be increased by 20% – which translates into additional cash flow of \$20,000 at the end of each year for five years. If $i=15\%$ per year, how much can we afford to invest to overhaul this machine?



Solution

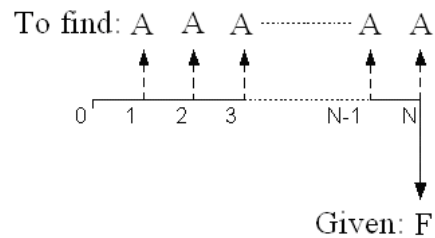
3.3522

$$P = \$20,000 (P/A, 15\%, 5) = \$67,044$$

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Finding A When Given F

Given a future sum of money, F , how might we determine the equivalent uniform end-of-period series, A , for N periods?



$$F = A \left[\frac{(1+i)^N - 1}{i} \right] \Rightarrow A = \frac{F}{\left[\frac{(1+i)^N - 1}{i} \right]} = F \left[\frac{i}{(1+i)^N - 1} \right]$$

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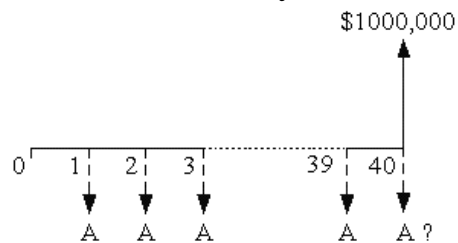
Uniform Series Sinking Fund Factor

- $\left[\frac{i}{(1+i)^N - 1} \right]$ is the *uniform series sinking fund factor*
- $\left[\frac{i}{(1+i)^N - 1} \right] = (A / F, i\%, N)$
- $A = F (A / F, i\%, N)$

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Example finding A given F

Recall that you would need to deposit \$22,100 today into an account paying 10% per year in order to have \$1,000,000 40 years from now. Instead of the single deposit, what uniform annual deposit for 40 years would also make you a millionaire?



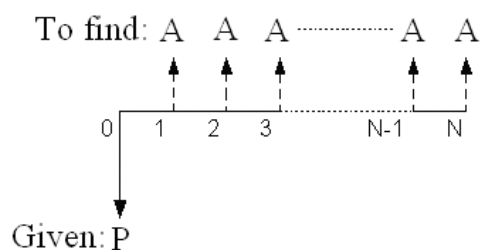
Solution

$$A = \$1,000,000 (A/F, 10\%, 40) = \$2,300$$

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Finding A When Given P

Given a present sum of money, P, how might we determine the equivalent uniform end-of-period series, A, for N periods?



$$P = A \left[\frac{(1+i)^N - 1}{i(1+i)^N} \right] \Rightarrow A = \frac{P}{\left[\frac{(1+i)^N - 1}{i(1+i)^N} \right]} = P \left[\frac{i(1+i)^N}{(1+i)^N - 1} \right]$$

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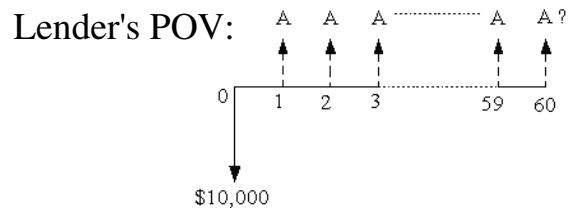
Uniform Series Capital Recovery Factor

- $\left[\frac{i(1+i)^N}{(1+i)^N - 1} \right]$ is the *uniform series capital recovery factor*
- $\left[\frac{i(1+i)^N}{(1+i)^N - 1} \right] = (A/P, i\%, N)$
- $A = P (A/P, i\%, N)$

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Example for finding A given P

Suppose you finance a \$10,000 car over 60 months at an interest rate of 1% per month. How much is your monthly car payment?



Solution

0.0222

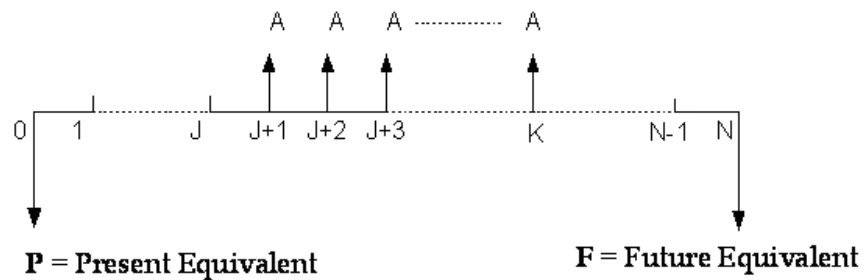
$$A = \$10,000 (A/P, 1\%, 60) = \$222 \text{ per month}$$

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Deferred Annuities

If we are looking for a present (future) equivalent sum at time other than one period prior to the first cash flow in the series (coincident with the last cash flow in the series) then we are dealing with a deferred annuity.

$A = \text{Uniform Amounts}$

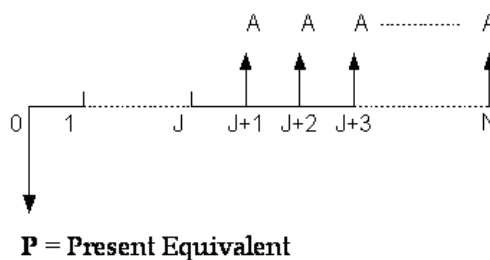


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Deferred Annuities for finding P

Dealing with a deferred annuity when looking for a present equivalent sum:

$A = \text{Uniform Amounts}$

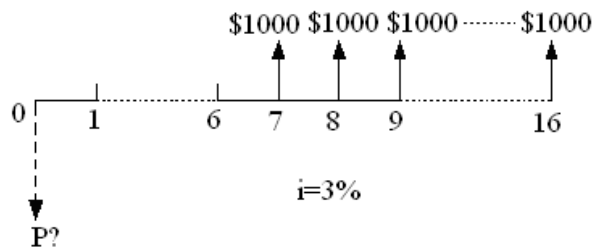


$$P = A (P/A, i\%, N - J)(P/F, i\%, J)$$

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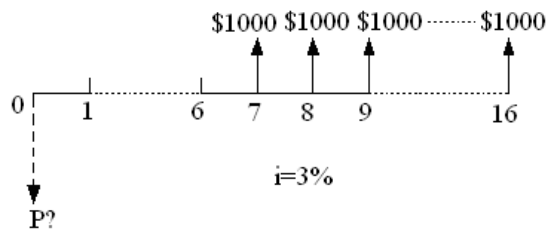
Example: Deferred Annuities for Present Equivalent Sum

How much do you need to deposit today into an account that pays 3% per year so that you can make 10 equal annual withdrawals of \$1,000, with the first withdrawal being made seven years from now?



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Solution



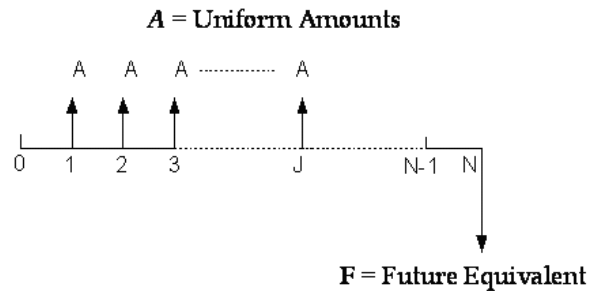
$$P_6 = \$1,000(P/A, 3\%, 10) = \$1,000(8.5302) = \$8,530.20$$

$$P_0 = P_6(P/F, 3\%, 6) = \$8,530.20(0.8375) = \$7,144$$

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Deferred Annuities for finding F

Dealing with a deferred annuity when looking for a future equivalent sum:

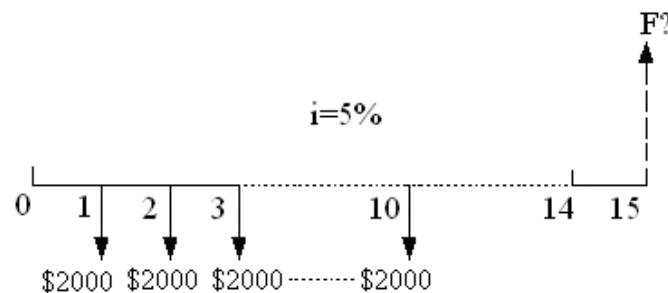


$$F = A(F/A, i\%, J)(F/P, i\%, N - J)$$

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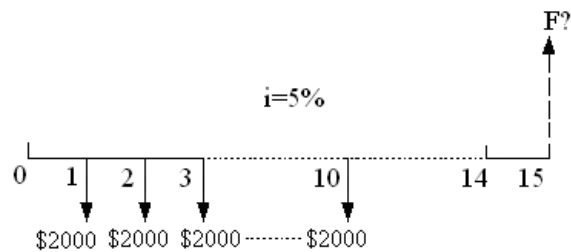
Example: Deferred Annuities for Future Equivalent Sum

Assume you make 10 equal annual deposits of \$2,000 into an account paying 5% per year. How much is in the account 5 years after the last deposit?



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Solution



$$F_{10} = \$2000(F/A, 5\%, 10) = \$2000(12.5779) = \$25,156$$

$$F_{15} = F_{10} (F/P, 5\%, 5) = \$25,156(1.2763) = \$32,106.6$$

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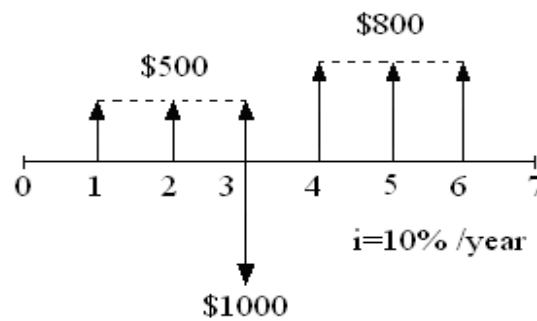
Multiple Interest Factors

Some situations include multiple unrelated sums or series, requiring the problem be broken into components that can be individually solved and then re-integrated.

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Example: Multiple Interest Factors

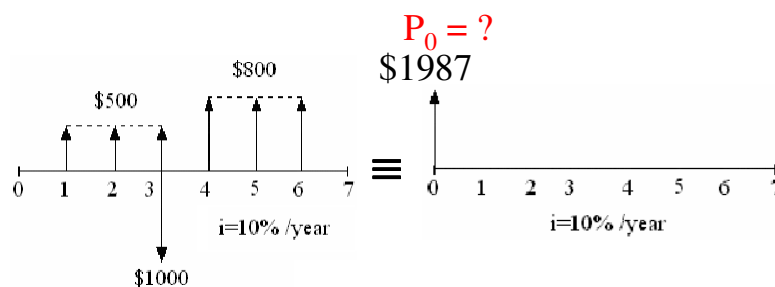
Given:



Find: P_0 , F_6 , F_7 , A_{1-6}

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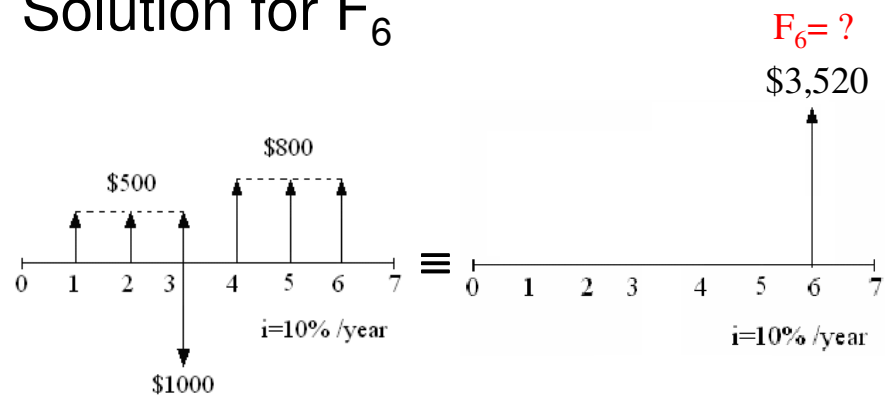
Solution for P_0



$$P_0 = 800 (P/A, 10\%, 3) \overset{0.7513}{(P/F, 10\%, 3)} + 500 \overset{2.4869}{(P/A, 10\%, 3)} - 1000 \overset{0.7513}{(P/F, 10\%, 3)} = \$1987$$

50

Solution for F_6



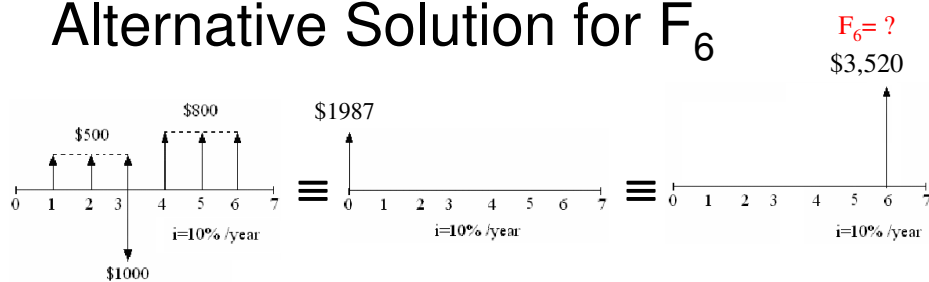
$$F_6 = 800 (F/A, 10\%, 3) + 500 (F/A, 10\%, 3)(F/P, 10\%, 3) - 1000 (F/P, 10\%, 3) = \$3,520$$

3.31
3.31
1.331

1.331

51

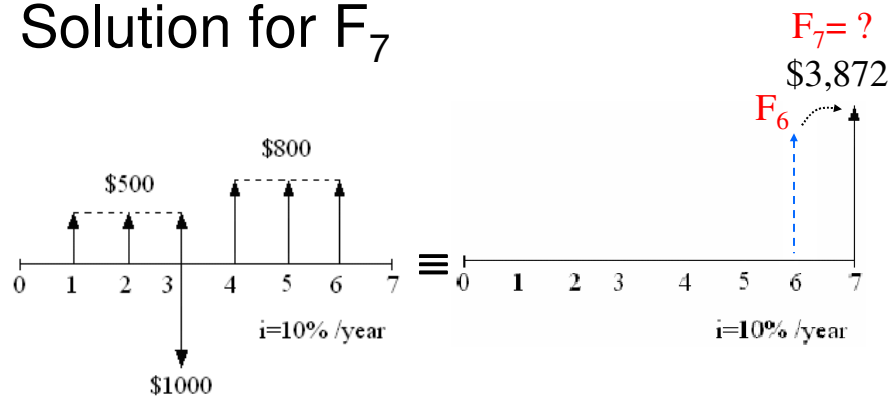
Alternative Solution for F_6



$$F_6 = \$1987 (F/P, 10\%, 6) = \$3,520$$

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Solution for F_7



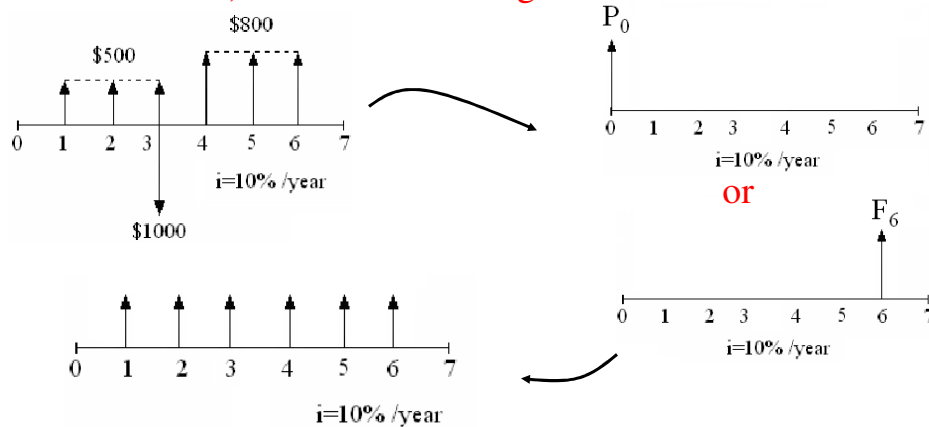
$$F_6 = 800 (F/A, 10\%, 3) + 500 (F/A, 10\%, 3)(F/P, 10\%, 3) - 1000 (F/P, 10\%, 3) = \$3,520$$

$$F_7 = 3,520 (F/P, 10\%, 1) = \$3,872$$

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Solution procedure for finding Annuity of a given multiple cash flows

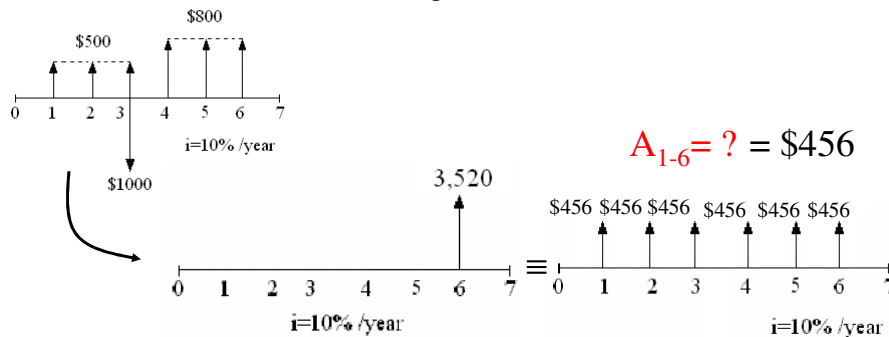
1) Convert into a single cash flow form



2) Convert into annuity form

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Solution for A_{1-6}



$$A_{1-6} = F_6 (A/F, 10\%, 6) = 3520 (0.1296) = \$456$$

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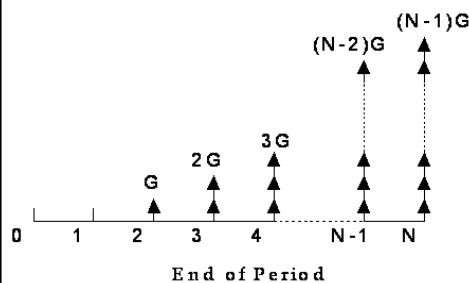
Uniform Gradient Series

- What if we have cash flows (revenues or expenses) that are projected to increase or decrease by a uniform amount each period? (e.g. maintenance costs, rental income)
- We call this a uniform gradient series (G)
- We can have positive or negative gradients if the slope of the cash flows is positive or negative, respectively

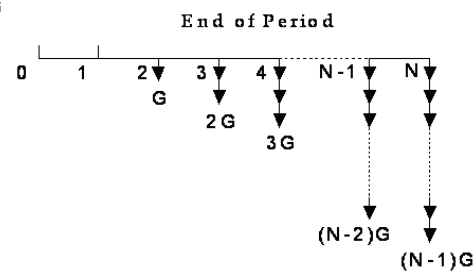
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CFDs of Uniform Gradient Series

Positive Gradient



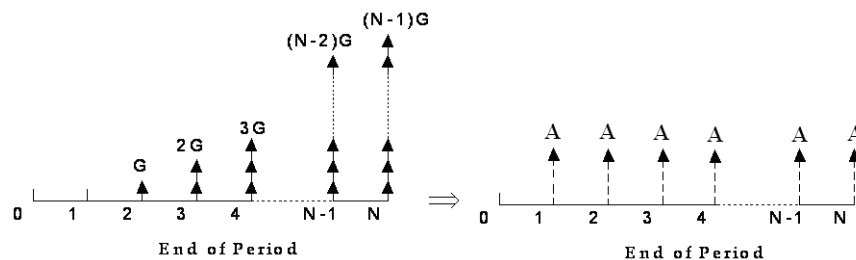
Negative Gradient



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Finding A When Given G

Given a uniform gradient series for N periods, G, how might we determine the equivalent uniform end-of-period series, A, for N periods?



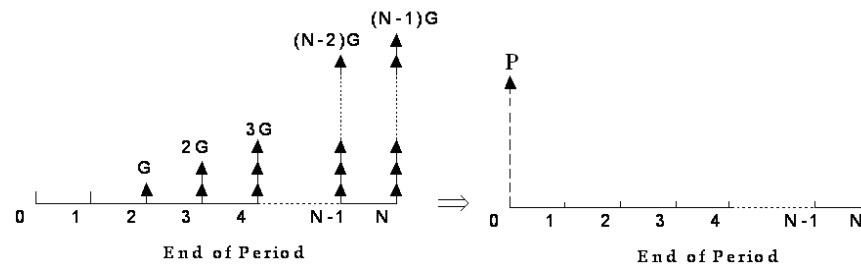
$$A = G \left[\frac{1}{i} - \frac{N}{(1+i)^N - 1} \right] = \text{gradient to uniform series conversion factor}$$

$$A = G (A / G, i\%, N)$$

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Finding P When Given G

Given a uniform gradient series for N periods, G, how might we determine the equivalent present sum at t=0?



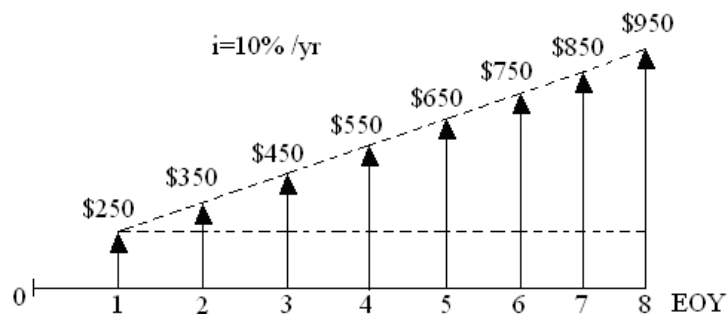
$$P = G \left\{ \frac{1}{i} \left[\frac{(1+i)^N - 1}{(1+i)^N} - \frac{N}{(1+i)^N} \right] \right\} = \text{gradient to present equivalent conversion factor}$$

$$P = G (P/G, i\%, N)$$

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Positive Gradient Example

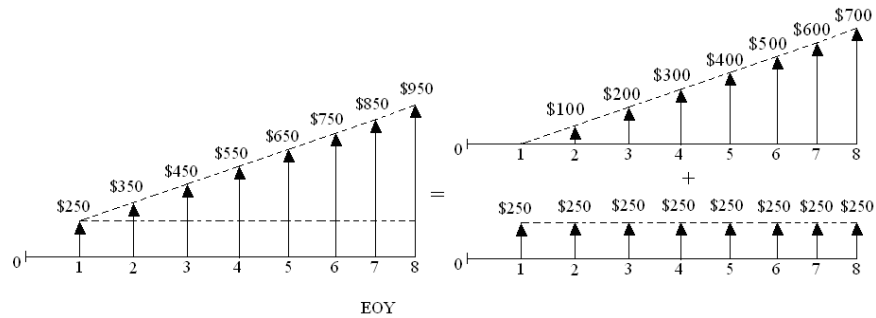
Given:



Find: P_0 and A_{1-8}

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Solution procedure

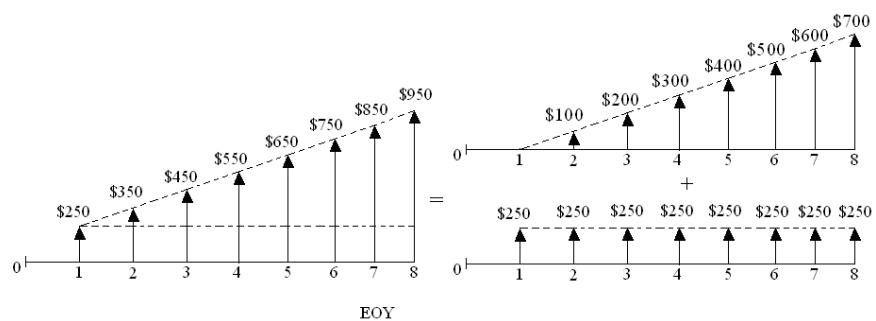


G = constant by which the cash flows increase or decrease each period.

$G = \$100$ in the above example.

61

Solution for P_0



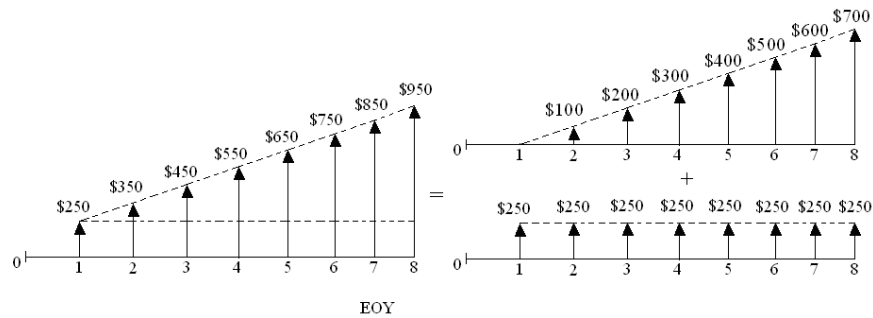
5.3349

16.029

$$P_0 = \$250 (P/A, 10\%, 8) + \$100 (P/G, 10\%, 8) = \$2,937$$

62

Solution for A_{1-8}



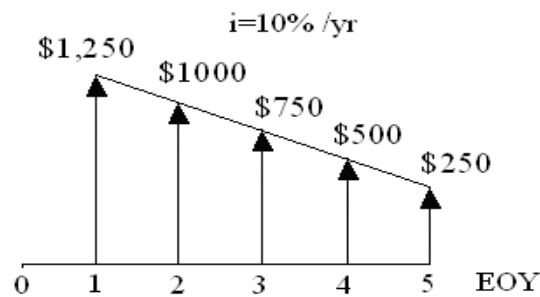
3.0045

$$A_{1-8} = \$250 + 100 (A/G, 10\%, 8) = \$550.45$$

63

Negative Gradient Example

Given:

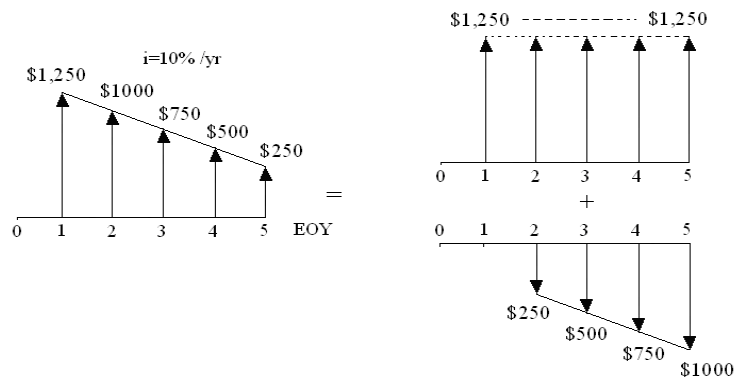


Find:

Equivalent values for these cash flows at: P_0 and A_{1-5}

64

Solution procedure

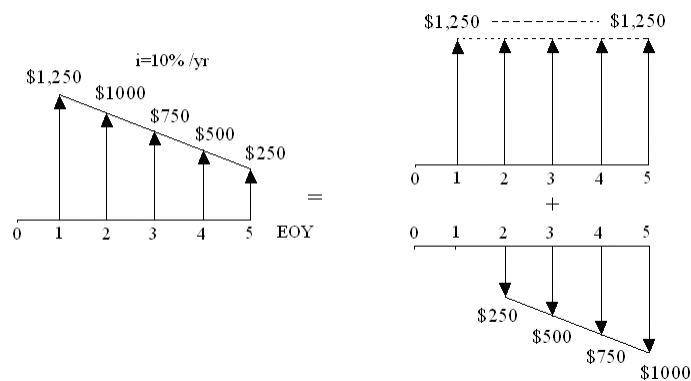


The gradient is negative (e.g., an increasing cost) and the uniform series is positive (e.g., steady stream of revenue).

$$G = -250 \quad \text{Underlying annuity (A)} = +1250$$

65

Solution for P_0



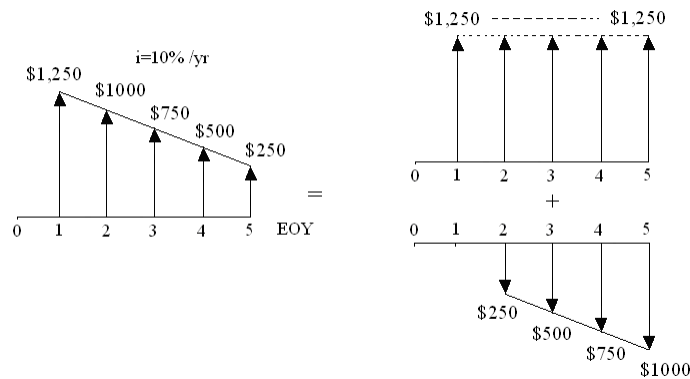
3.7908

6.86

$$P_0 = 1,250(P/A, 10\%, 5) - 250(P/G, 10\%, 5) = \$3,023.5$$

66

Solution for A_{1-5}



1.8101

$$A_{1-5} = 1,250 - 250(A/G, 10\%, 5) = \$797.48$$

67

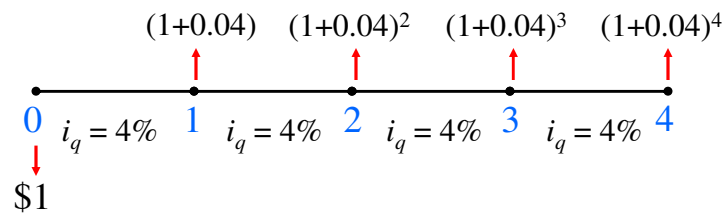
Nominal and Effective Interest Rates

- Nominal interest (r) = interest compounded more than one interest period per year but quoted on an annual basis.
- Example: 16%, compounded quarterly
- Effective interest (i) = actual interest rate earned or charged for a specific time period.
- Example: $16\%/4 = 4\%$ effective interest for each of the four quarters during the year.

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Nominal and Effective Interest Rates

$r = 16\%$ /year nominal interest rate /year
compounded quarterly



$$(1+0.04)^4 = (1+0.17)$$

$i = 17\%$ effective interest rate /year

69

Effective interest rate

$$i = \left(1 + \frac{r}{M}\right)^M - 1$$

where

i = effective annual interest rate

r = nominal interest rate per year

M = number of compounding periods per year

r / M = interest rate per interest period

70

Example 2.1

Find the effective interest rate per year at a nominal rate of 18% compounded (1) quarterly, (2) semiannually

(1) Quarterly compounding

$$i = \left(1 + \frac{0.18}{4}\right)^4 - 1 = 0.1925 = 19.25\% \text{ /yr}$$

(2) Semiannually compounding

$$i = \left(1 + \frac{0.18}{2}\right)^2 - 1 = 0.1881 = 18.81\% \text{ /yr}$$

71

Example 2.2

Nominal interest rate is 18% compounded monthly.

Interpretation: This is a normal statement of an interest rate where the related time period is one year, and the sub-period is one month.

$r = 18\%$; $M = 12$; $i_M = 1\frac{1}{2}\%$;

What if $N=3$ years and $P = \$1,000$, find F .

72

Example 2.2 (solution)

$$i = (1 + 0.015)^{12} - 1 = 0.1956 = 19.56\%$$

$$F = 1000(1 + 0.1956)^3 = \$1709.14$$

or

$$i_M = 1.5\% \text{ /month and } N = 3 \times 12 = 36 \text{ months}$$

$$F = 1000(1 + 0.015)^{36} = \$1709.14$$

73

Example 2.3

A credit card company advertises an A.P.R. of 16.9% compounded daily on unpaid balances. What is the effective interest rate per year being charged?

Solution

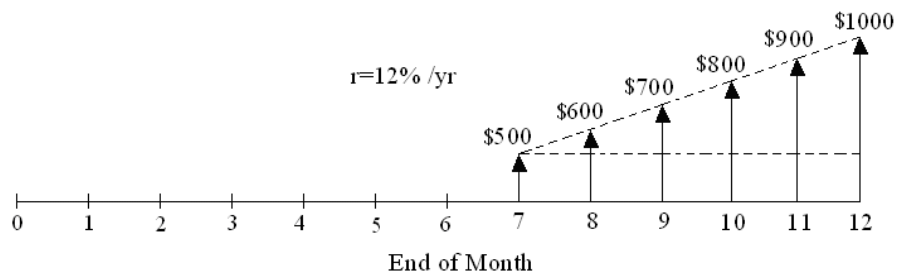
$$r = 16.9\% \text{ and } M = 365$$

$$i_{eff} = \left(1 + \frac{0.169}{365}\right)^{365} - 1 = 0.184 = 18.4\%$$

74

Example 2.4

Find P_0 when $r = 12\%$, compounded monthly



75

Example 2.4 (solution)

We have monthly cash flows so we need to use a monthly interest rate

$$i / \text{mo.} = r / M = 12\% / 12 = 1\% \text{ per month}$$

$$\begin{aligned} P_0 &= [\$500 (P/A, 1\%, 6) + \$100 (P/G, 1\%, 6)] (P/F, 1\%, 6) \\ &= [\$500 (5.7955) + \$100 (14.321)] (0.9420) = \$4078.7 \end{aligned}$$

76

Two situations for given r per year and M .

(1) Cash flows are annual

Procedure: Find i/yr .

$$i = \left(1 + \frac{r}{M}\right)^M - 1 \quad \text{and discount/compound annual cash flows at } i/\text{yr}.$$

(2) Cash flows occur M times per year

Procedure: Find the interest rate that corresponds to M , which is r/M per time period (e.g., quarter, month)

Then discount/compound the M cash flows per year at r/M for the time period given

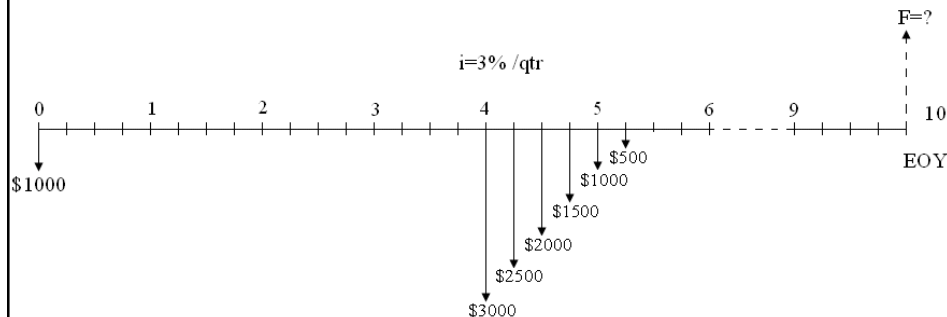
77

Example 2.5

If you deposit \$1,000 now, \$3,000 four years from now followed by five quarterly deposits decreasing by \$500 per quarter at an interest rate of 12% per year compounded quarterly, how much money will you have in your account 10 years from now?

78

Example 2.5 (solution)



$r/M = 0.12 / 4 = 3\%$ per quarter Year 3.75 = 15th Quarter

$$P \text{ @yr. 3.75} = P \text{ @qtr. 15} = 3000(P/A, 3\%, 6) - 500(P/G, 3\%, 6) \\ = \$9713.60$$

$$F \text{ @yr. 10} = F \text{ @qtr. 40} = 9713.60(F/P, 3\%, 25) + 1000(F/P, 3\%, 40) \\ = \mathbf{\$23,600.34}$$

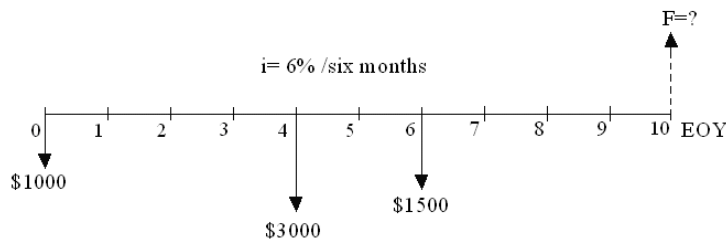
79

Example 2.6

If you deposit \$1,000 now, \$3,000 four years from now, and \$1,500 six years from now at an interest rate of 12% per year compounded semiannually, how much money will you have in your account 10 years from now?

80

Example 2.6 (solution)



$$i \text{ per year} = \left(1 + \frac{0.12}{2}\right)^2 - 1 = 0.1236$$

$$F = \$1,000(1+0.1236)^{10} + \$3,000(1+0.1236)^6 + \$1,500(1+0.1236)^4$$

$$= \$11,634.50 \quad \text{or}$$

$r/M = 6\%$ per half-year and $N = 10 \times 2 = 20$ six months

$$F = 1000(F/P, 6\%, 20) + 3000(F/P, 6\%, 12) + 1500(F/P, 6\%, 8)$$

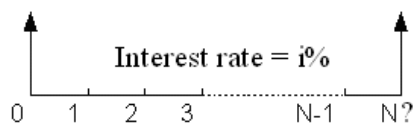
$$= \$11,634.50$$

81

Finding N Given P , F and i

P = Present Equivalent

F = Future Equivalent



$$F = P (1+i)^N$$

or

$$F = P (F/P, i\%, N)$$

$$N \log (1+i) = \log (F/P)$$

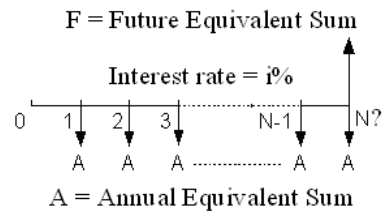
$$(F/P, i\%, N) = F/P$$

$$N = \log (F/P) / \log (1+i)$$

N is found from
interest table

82

Finding N Given F , A and i



$$N = \frac{\log\left(\frac{F}{A} + \frac{1}{i}\right) + \log(i)}{\log(1+i)}$$

or

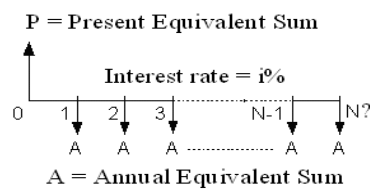
$$F = A (F/A, i\%, N)$$

$$(F/A, i\%, N) = F/A$$

N is found from
interest table

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Finding N Given P , A and i



$$N = (-1) \left[\frac{\log\left(\frac{1}{i} - \frac{P}{A}\right) + \log(i)}{\log(1+i)} \right]$$

or

$$P = A (P/A, i\%, N)$$

$$(P/A, i\%, N) = P/A$$

N is found from
interest table

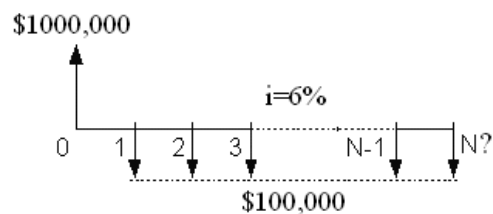
84

Example 2.7

Suppose that your rich uncle has \$1000,000 that he wishes to distribute to his heirs at the rate of \$100,000 per year. If the \$1000,000 is deposited in a bank account that earns 6% interest per year, how many years will it take to completely deplete the account?

85

Example 2.7 (solution)



$$N = (-1) \left[\frac{\log\left(\frac{1}{i} - \frac{P}{A}\right) + \log(i)}{\log(1+i)} \right] \Rightarrow N = (-1) \left[\frac{\log\left(\frac{1}{0.06} - 10\right) + \log(0.06)}{\log(1.06)} \right]$$

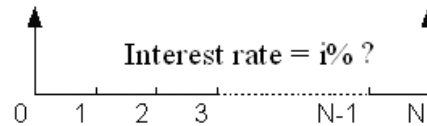
$$N = 15.7 \text{ years} \cong 16 \text{ years}$$

86

Finding i , Given P , F and N

P = Present Equivalent

F = Future Equivalent



$$F = P(1+i)^N$$

$$(1+i)^N = F / P$$

$$i = \sqrt[N]{\frac{F}{P}} - 1$$

87

Example 2.8

Suppose you borrowed \$8,000 now and the lender wants \$11,713 to be paid 4 years from now, what is the interest rate accrued on your debt?

Solution

$$11713 = 8000 (1+i)^4$$

$$(1+i) = \sqrt[4]{\frac{11713}{8000}} = 1.1$$

$$i = 1.1 - 1 = 0.1 = 10\%$$

88

Interest rates that vary with time

Find the present equivalent value given a future value and a varying interest rate over the period of the loan

$$P = \frac{F}{\prod_{k=1}^N (1 + i_k)}$$

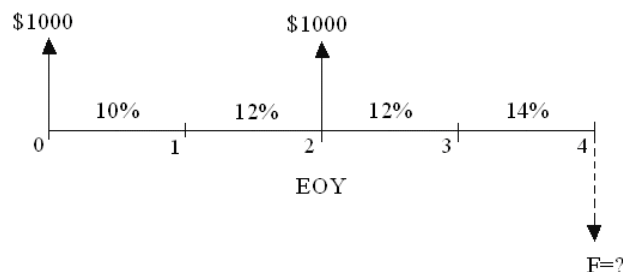
Using single (average) interest rate for compounding / discounting the single cash flows:

$$(1+i)^N = \prod_{k=1}^N (1+i_k) \quad \Rightarrow \quad i = \sqrt[N]{\prod_{k=1}^N (1+i_k)} - 1$$

89

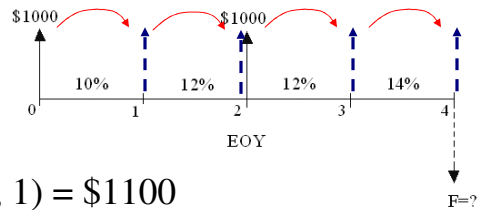
Example 2.9

A person has made an arrangement to borrow \$1000 now and another \$1000 two years hence. The entire obligation is to be paid at the end of four years. If the projected interest rates in years one, two, three and four are 10%, 12%, 12%, and 14%, respectively, how much will be repaid as a lump-sum amount at the end of four years?



90

Example 2.9 (solution)



$$F_1 = 1000(F/P, 10\%, 1) = \$1100$$

$$F_2 = 1100(F/P, 12\%, 1) + 1000 = \$2,232$$

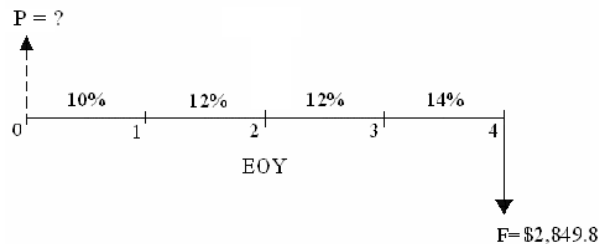
$$F_3 = 2232(F/P, 12\%, 1) = \$2,500$$

$$F_4 = 2500(F/P, 14\%, 1) = \$2,849.8$$

91

Example 2.10

Given F_4 and interest rates that vary with time, find P_0

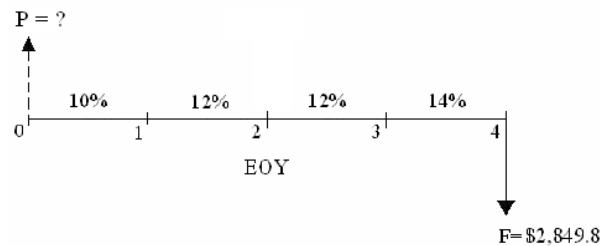


Solution

$$P = \frac{F}{\prod_{k=1}^N (1+i_k)} = \frac{2849.8}{(1+0.1)(1+0.12)^2(1+0.14)} = \$1811.68$$

92

Example 2.10 (alternative solution)



$$i = \sqrt[N]{\prod_{k=1}^N (1 + i_k)} - 1 \rightarrow i = \sqrt[4]{1.5730176} - 1 = 0.1199$$

$$\prod_{k=1}^N (1 + i_k) = (1 + 0.1)(1 + 0.12)^2(1 + 0.14) = 1.5730176$$

$$P_0 = 2849.8(1 + 0.1199)^{-4} = \$1811.74$$

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Continuous compounding and discrete cash flows

Continuous compounding assumes cash flows occur at discrete intervals, but compounding is continuous throughout the interval.

Given r (nominal per year interest rate) and M (compounding per year)

$$\text{one unit of principal} = [1 + (r / M)]^M = e^r$$

Continuous compounding: $M \rightarrow \infty$

$$\text{Given } M / r = p \rightarrow [1 + (r / M)]^M = [1 + (1/p)]^p = e^r$$

$$\lim_{p \rightarrow \infty} [1 + (1/p)]^p = e^1 = 2.71828$$

$$[1 + (r / M)]^M = (1 + i) = e^r \rightarrow i = e^r - 1$$

94

Finding F given P

$$F = P (1+i)^N = P (1 + e^r - 1)^N = P e^{rN}$$

e^{rN} is continuous compounding compound amount

Functionally expressed as (F / P, $\underline{r}\%$, N)

Predetermined values are in appendix D of text

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Continuous compounding interest factors

Conversion	Interest Factor	Functional Symbol
Finding P given F	e^{-rN}	(P / F, $\underline{r}\%$, N)
Finding F given A	$(e^{rN} - 1) / (e^r - 1)$	(F / A, $\underline{r}\%$, N)
Finding A given F	$(e^r - 1) / (e^{rN} - 1)$	(A / F, $\underline{r}\%$, N)
Finding P given A	$(e^{rN} - 1) / (e^{rN}) (e^r - 1)$	(P / A, $\underline{r}\%$, N)
Finding A given P	$e^{rN} (e^r - 1) / (e^{rN} - 1)$	(A / P, $\underline{r}\%$, N)

96

Example 2.11

Suppose that one has a present loan of \$1000 and desires to determine what equivalent uniform end of year payments, A , could be obtained from it for 10 years if the nominal interest rate is 20% compounded continuously ($M=\infty$).

Solution

$$A = P (A/P, \underline{20\%}, 10) = \$1000 (0.25605) = \$256$$

Answer to the same problem with discrete compounding ($M=1$)

$$A = \$1000(A/P, 20\%, 10) = \$1000 (0.2385) = \$238.5$$

97

Example 2.12

An individual needs \$12000 immediately as a down payment on a new home. Suppose that he can borrow this money from his insurance company. He must repay the loan in equal payments every six months over the next eight years. The nominal interest rate being charged is 7% compounded continuously. What is the amount of each payment?

Solution

$$r \text{ (nominal interest rate / six months)} = 7\% / 2 = 3.5\%$$

$$N \text{ (number of cash flows (periods))} = 8 \times 2 = 16 \text{ six months}$$

Notice that compounding is still infinite during each six months

$$A = P (A/P, \underline{3.5\%}, 16) = 12000 (0.08307) = \$997$$

98